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The Role of Initial Conditions in Presence of Extra Dimensions

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Abstract

Quantum field theory in 4+1 dimensional bulk space with boundary, representing a 3-brane, is considered. We study the impact of the initial conditions in the bulk on the field dynamics on the brane. We demonstrate that these conditions determine the Kaluza-Klein measure. We also establish the existence of a rich family of quantum fields on the brane, generated by the same bulk action, but corresponding to different initial conditions. A simple classification of these fields is proposed and it is shown that some of them lead to ultraviolet finite theories, which have some common features with strings.

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1 Introduction

There have been in the past various attempts to render the short distance properties of quantum fields milder by modifying or relaxing the requirement of local commutativity. The first efforts in this direction can be traced back to the late forties [1, 2] and early fifties [3, 4], when such relevant ideas like quantized space-time, nonlocal formfactors and the concept of weak local commutativity [5] already emerged. The modern approach to the subject is dominated by string theory and the deeply related field theory on noncommutative space-time. In this paper we discuss locality in the light of the recent developments in quantum field theory with extra dimensions and branes [6] - [11], focusing mainly on the role of the initial conditions.

In order to fix the ideas, we consider a toy model in five-dimensional bulk space $\mathcal{M} = \mathbb{R}^4 \times \mathbb{R}_+$, where \mathbb{R}_+ is the half line $\{y \in \mathbb{R} : y > 0\}$. We adopt the coordinates $(x, y) \in \mathbb{R}^4 \times \mathbb{R}_+$ with $x \equiv (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x})$ and a diagonal flat metric $G_{\alpha\beta}$ ($\alpha, \beta = 0, \dots, 4$) with $\text{diag } G = (1, -1, -1, -1, -1)$. The boundary ∂M coincides with the 3 + 1-dimensional Minkowski space \mathbf{M}_{3+1} , modeling a static 3-brane. The dynamics of the model is defined by the action

$$S = \frac{1}{2} \int_{-\infty}^{\infty} d^4x \int_0^{\infty} dy \left(\partial^\alpha \Phi \partial_\alpha \Phi - M^2 \Phi^2 \right) (x, y) - \int_{-\infty}^{\infty} d^4x \left(\frac{\eta}{2} \Phi^2 - \frac{g}{3!} \Phi^3 \right) (x, 0), \quad (1)$$

whose variation gives the equation of motion

$$(\partial^\alpha \partial_\alpha + M^2) \Phi(x, y) = 0, \quad y > 0, \quad (2)$$

and the boundary condition

$$\left[(\partial_y - \eta) \Phi + \frac{g}{2} \Phi^2 \right] (x, 0) = 0. \quad (3)$$

Being localized on the boundary, the interaction generates a nonlinear boundary condition. We will work out explicitly the case $\eta \geq 0$, commenting at the end of the paper on the range $\eta < 0$. The action (1) breaks down all translations, rotations and Lorentz transformations involving the y -coordinate. Notice, however, that brane Poincaré invariance is preserved.

Eqs.(2,3) define an initial boundary value problem. Accordingly, the solution depends on the initial conditions fixed on a time-like hypersurface in \mathcal{M} . In the context of quantum field theory these conditions are determined by the equal-time commutation relations. The general form of the nontrivial commutator in our case is

$$[(\partial_0 \Phi)(x^0, \mathbf{x}_1, y_1), \Phi(x^0, \mathbf{x}_2, y_2)] = -i\delta(\mathbf{x}_1 - \mathbf{x}_2) \xi(y_1, y_2), \quad y_1, y_2 > 0. \quad (4)$$

The distribution ξ plays an essential role in what follows. The basic physical requirements of positivity and locality provide some restrictions on ξ , but do not fix ξ completely. The asymmetry of \mathcal{M} , caused by the extra dimension $y \in \mathbb{R}_+$, implies the existence of inequivalent classes of ξ -distributions, inducing different ultraviolet (UV) behavior on the brane. Our main goal below will be to establish and investigate this new phenomenon, which has no counterpart in local relativistic Lagrangian field theory on \mathbf{M}_{3+1} . We will show that the dynamics on the brane \mathbf{M}_{3+1} strongly depends on the initial conditions in the bulk \mathcal{M} .

Let us analyze first the impact of positivity on ξ . Treating the interaction perturbatively in g , we start with the case $g = 0$. We first introduce the complete orthonormal set of eigenfunctions ($\lambda > 0$)

$$\psi(y, \lambda) = e^{-i\lambda y} + B(\lambda) e^{i\lambda y}, \quad B(\lambda) \equiv \frac{\lambda - i\eta}{\lambda + i\eta}, \quad (5)$$

of the operator $-\partial_y^2$ on \mathbb{R}_+ , whose domain is defined in agreement with (3) by the boundary condition

$$\lim_{y \downarrow 0} (\partial_y - \eta) \psi(y) = 0. \quad (6)$$

Then $\xi(y_1, y_2)$ is generated by a distribution σ of the single variable $\lambda \in \mathbb{R}_+$ via

$$\xi(y_1, y_2) = \int_0^\infty \frac{d\lambda}{2\pi} \bar{\psi}(y_1, \lambda) \sigma(\lambda) \psi(y_2, \lambda). \quad (7)$$

The requirement of positivity of the metric in the state space takes a simple form in terms of σ and reads

$$\sigma(\lambda) \geq 0 \quad (8)$$

on \mathbb{R}_+ . We assume (8) in what follows and proceed by clarifying the physical content of σ . It is convenient for this purpose to compute the two-point vacuum expectation value (Wightman function) of Φ . The result is [12]

$$\langle \Phi(x_1, y_1) \Phi(x_2, y_2) \rangle_0 = \int_0^\infty \frac{d\lambda}{2\pi} \bar{\psi}(y_1, \lambda) \sigma(\lambda) \psi(y_2, \lambda) W_{M^2 + \lambda^2}(x_{12}), \quad (9)$$

where

$$W_{m^2}(x) = \int_{-\infty}^\infty \frac{d^4 p}{(2\pi)^4} e^{-ipx} \theta(p^0) 2\pi \delta(p^2 - m^2) \quad (10)$$

is the familiar two-point scalar function of mass m^2 in \mathbf{M}_{3+1} . It is perhaps useful to recall that the integral over λ in (9) and in what follows must be interpreted in the sense of distributions, i. e. after smearing the integrand with suitable test functions over \mathcal{M} . The choice of the test function space depends on the behavior of σ for $\lambda \rightarrow \infty$ and will be specified below.

Eq.(9) completely fixes the correlation functions of Φ for $g = 0$. The latter determine the quantum field φ , induced on the brane, by means of

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_0 = \lim_{y_i \downarrow 0} \langle \Phi(x_1, y_1) \cdots \Phi(x_n, y_n) \rangle_0. \quad (11)$$

The limit (11) defines integral representations for the correlation functions of φ , whose fundamental features are encoded in the two-point function

$$w(x_{12}) = \langle \varphi(x_1) \varphi(x_2) \rangle_0 = \int_0^\infty d\lambda^2 \varrho(\lambda^2) W_{\lambda^2}(x_{12}), \quad x_{12} \equiv x_1 - x_2, \quad (12)$$

with

$$\varrho(\lambda^2) = \theta(\lambda^2 - M^2) \frac{\sqrt{\lambda^2 - M^2} \sigma(\sqrt{\lambda^2 - M^2})}{\pi(\lambda^2 + \eta^2 - M^2)}. \quad (13)$$

We observe in passing that the expansion of ϱ around $\eta = 0$ produces spurious divergences at $\lambda^2 = M^2$ in (12). This feature represents a serious obstruction to the perturbative treatment [13] of the boundary term $\frac{\eta}{2} \Phi^2$ in the action (1). Using (10), the integral in (12) can be easily performed in momentum space and for the Fourier transform \widehat{w} of w one finds

$$\widehat{w}(p) = 2\theta(p^0) \frac{\theta(p^2 - M^2) \sqrt{p^2 - M^2}}{p^2 - M^2 + \eta^2} \sigma(\sqrt{p^2 - M^2}). \quad (14)$$

Eqs.(12-14) establish the physical meaning of σ . We deduce that the mass spectrum of φ belongs to $[M^2, \infty)$. The corresponding eigenstates represent the Kaluza-Klein (KK) modes, σ defining the weight each KK mode contributes to the field φ . The initial conditions in the bulk determine therefore the KK measure ϱ on the brane. This result provides valuable information about a central problem in quantum field theory with extra dimensions - the integration over the KK modes. In fact, the KK measure is an essential element for the complete understanding of this recently debated problem.

In order to classify the possible initial conditions (KK measures), one can adopt the concept of local commutativity. Let us introduce for this purpose some terminology. We will say that an initial condition (4) is of type A, if the corresponding bulk field Φ is local.

Otherwise, we call the initial condition of type B. One can further refine this classification using the locality properties of φ . Since in passing from the bulk to the brane one is suppressing a spatial dimension, type A initial conditions generate only local brane fields. This is not the case for type B conditions. In fact, there exist [12] nonlocal bulk fields which induce local brane fields. We call the relative initial conditions of type B₁. Finally, if both Φ and φ are nonlocal, one has type B₂ initial conditions. We shall illustrate below this elementary classification, focusing in each case on the UV behavior of φ .

Consider first type A initial conditions. Since the bulk hypersurface $x^0 = \text{const.}$ is space-like, local commutativity of Φ applied to (4) implies that the support of ξ is in $y_1 = y_2$. A large class of such distributions is obtained by setting

$$\sigma(\lambda) = P(\lambda^2), \quad (15)$$

where, according to (8), P is any positive definite polynomial. In fact, from (7) one gets

$$\xi(y_1, y_2) = P(-\partial_{y_1}^2) \delta(y_{12}), \quad (16)$$

which has the required support property and generates a local bulk field Φ . This feature can be checked directly, taking into account [14, 15] that the signals propagating in the bulk are also reflected from the boundary, the reflection coefficient being $B(\lambda)$.

To our knowledge only the case

$$P(\lambda^2) = \text{const.} > 0 \quad (17)$$

has been explored till now in the literature. With this choice the propagator

$$\tau(x) \equiv i\theta(x^0)w(x) + i\theta(-x^0)w(-x) \quad (18)$$

has the milder UV behavior within the set of initial conditions defined by eqs.(4,16). Indeed, using (14) one gets

$$\hat{\tau}(p) = -\frac{\sqrt{M^2 - p^2} - \eta}{p^2 - M^2 + \eta^2 + i\varepsilon} P(p^2 - M^2). \quad (19)$$

Because of positivity, the optimal decay property of the right hand side of (19) for large Euclidean momenta $p_E = (-ip^0, \mathbf{p})$ is obtained in the case (17). This is the only motivation we are aware for selecting (17) among all positive definite polynomials. Eq.(19) implies also that the boundary interaction ($g \neq 0$) in (1) is nonrenormalizable for all initial conditions of the form (4,16). Moreover, the relative brane fields φ are not canonical.

Concerning the type B initial conditions, we start with the simple example

$$\sigma(\lambda) = \lambda, \quad \eta = 0. \quad (20)$$

One easily derives (Pv stands for principal value)

$$\xi(y_1, y_2) = -\frac{1}{\pi} \left(\text{Pv} \frac{1}{y_{12}^2} + \text{Pv} \frac{1}{\widetilde{y}_{12}^2} \right), \quad \widetilde{y}_{12} \equiv y_1 + y_2, \quad (21)$$

whose support is not localized at $y_1 = y_2$. Therefore, the corresponding bulk field Φ is nonlocal. In spite of this fact, eqs.(12,13) give

$$\langle \varphi(x_1) \varphi(x_2) \rangle_0 = \frac{1}{\pi} \int_0^\infty d\lambda^2 \theta(\lambda^2 - M^2) W_{\lambda^2}(x_{12}), \quad (22)$$

which defines a local generalized free field [16] on the brane. Thus, the $y \rightarrow 0$ limit absorbs all nonlocal effects present in the bulk and, according to our classification, the initial condition (20) is of type B₁. Such conditions have been discovered and described in some detail in [12]. A remarkable subset of the class B₁ is obtained by requiring

$$\int_0^\infty d\lambda^2 \varrho(\lambda^2) \equiv C < \infty. \quad (23)$$

Besides of being local, the field φ induced on the brane is in this case also canonical, namely

$$[(\partial_0 \varphi)(x^0, \mathbf{x}_1), \varphi(x^0, \mathbf{x}_2)] = -i C \delta(\mathbf{x}_1 - \mathbf{x}_2). \quad (24)$$

The bound (23) implies also that the brane propagator satisfies

$$p_E^2 \widehat{\tau}(p_E) \leq C. \quad (25)$$

Thus $\widehat{\tau}(p_E)$ decays at least like $1/p_E^2$ when $p_E^2 \rightarrow \infty$, leading to a renormalizable perturbative expansion for the correlation functions of our model (1) on the brane. It has been shown in [12], that this UV behavior cannot be improved further within the class B₁. We conclude therefore that relaxing local commutativity in the bulk, but keeping this requirement on the brane, allows to generate at most renormalizable brane interactions. Bulk fields of the type B₁ can be constructed [12] on an AdS background as well and we expect that they may help for evading some no-go theorems [17], concerning the construction of Randall-Sundrum compactifications from supergravity.

Conformal invariance on the brane deserves also a special comment. Conformal covariant fields with anomalous dimension $d > 1$ are generated for $M = \eta = 0$ by initial conditions defined by

$$\sigma(\lambda) = \frac{\pi}{\Gamma(d-1)} \lambda^{2d-3}. \quad (26)$$

Except for $d = k + \frac{3}{2}$ with $k = 1, 2, \dots$, all these fields are induced by nonlocal bulk fields, which confirms the relevance of giving up local commutativity in the bulk.

A common feature of type A and type B₁ initial conditions is that both of them can be characterized by requiring σ to be polynomially bounded. As test functions in (9,12) one can use smooth functions of rapid decrease belonging to the Schwartz test function space \mathcal{S} [18]. Accordingly, the correlation functions are tempered distributions.

We turn now to type B₂ initial conditions. Generalizing eq.(16), we consider the series

$$\xi(y_1, y_2) = \sum_{n=0}^{\infty} \frac{c_n}{(2n)!} (-\partial_{y_1}^2)^n \delta(y_1 - y_2), \quad (27)$$

with a restriction on the growth of the coefficients c_n for $n \rightarrow \infty$. Let us assume for instance that

$$\overline{\lim}_{n \rightarrow \infty} |c_n|^{1/n} n^{-a} < \infty, \quad (28)$$

for some $0 < a < 2$. Such kind of distributions have been already applied in nonlocal quantum field theory by Efimov [19, 20]. Since the corresponding σ may grow exponentially for $\lambda \rightarrow \infty$, the test functions in momentum space must be of compact support. Consequently [18], the correlators in coordinate space define analytic functionals instead of tempered distributions. Referring for the details about such functionals to the mathematical literature (see e.g. [18]), we prefer to concentrate below on the explicit example

$$\sigma(\lambda) = e^{l^2 \lambda^2}, \quad (29)$$

where $l \geq 0$ is a free parameter with dimension of length. Notice that $l = 0$ reproduces the conventional initial condition $\xi(y_1, y_2) = \delta(y_{12})$. The requirement (28) is satisfied for $1 \leq a < 2$ and inserting (29) in (14), one gets the two-point function

$$\widehat{w}(p) = 2\theta(p^0) \frac{\theta(p^2 - M^2) \sqrt{p^2 - M^2}}{p^2 - M^2 + \eta^2} e^{l^2(p^2 - M^2)}, \quad (30)$$

carrying the complete information about φ for $g = 0$. As expected, both Φ and φ are nonlocal.

Switching on the interaction, one can analyze the model (1) in perturbation theory. From eqs.(18,30) one obtains the propagator

$$\widehat{\tau}(p) = -\frac{\sqrt{M^2 - p^2} - \eta}{p^2 - M^2 + \eta^2 + i\varepsilon} e^{l^2(p^2 - M^2)}. \quad (31)$$

We see that $\widehat{\tau}(p_E)$ decays exponentially for $p_E^2 \rightarrow \infty$, which implies that φ has UV finite perturbative expansion. In order to get an idea about this expansion, let us derive the one-loop self energy $\Sigma^{(1)}$. In the case $\eta = 0$ one has

$$\Sigma^{(1)}(p) = ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\sqrt{(M^2 - k^2)[M^2 - (p - k)^2]}}{(k^2 - M^2 + i\varepsilon)[(p - k)^2 - M^2 + i\varepsilon]} e^{l^2[k^2 + (p - k)^2 - 2M^2]}. \quad (32)$$

The integration in k can be performed by means of the integral representation

$$\frac{\sqrt{D}}{D + i\varepsilon} = \frac{1}{\sqrt{i\pi}} \int_0^\infty d\alpha \frac{e^{i\alpha(D + i\varepsilon)}}{\sqrt{\alpha}}, \quad \varepsilon > 0, \quad D \in \mathbb{R}. \quad (33)$$

One finds

$$\Sigma^{(1)}(p) = \frac{ig^2}{16\pi^3} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \frac{e^{-\varepsilon(\alpha_1 + \alpha_2)}}{\sqrt{\alpha_1 \alpha_2}} F(\alpha_1 - il^2, \alpha_2 - il^2), \quad (34)$$

with

$$F(\alpha_1, \alpha_2) = \frac{e^{ip^2 \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} - iM^2(\alpha_1 + \alpha_2)}}{(\alpha_1 + \alpha_2)^2}. \quad (35)$$

The effect of the exponential factor in the propagator (31) is the il^2 shift in both arguments of F in (34). Standard manipulations lead at this stage to the estimate ($\varepsilon \rightarrow 0$)

$$|\Sigma^{(1)}(p)| < \frac{g^2}{l^2} e^{l^2(p^2 - 2M^2)}, \quad (36)$$

which confirms that $\Sigma^{(1)}$ is finite for $l \neq 0$. The argument can be easily extended to the one-loop vertex function $\Lambda^{(1)}$ and applied to any order in perturbation theory. The key point is obviously the exponential factor in the propagator (31). It is instructive to recall in this respect that a similar mechanism works in string field theory as well. Indeed, the string vertices involve [21, 22] the factor $\exp(-\alpha' p_E^2)$, causing loops to converge. In spite of the fact that in our case such a factor enters the propagator, the net result is the same: the perturbative expansion of (1), with initial conditions defined by (4,7,29), is UV finite

order by order. This result shows that by an appropriate choice of the initial conditions in the bulk, one can reproduce some features of string theory in the context of quantum field theory with extra dimensions. In this spirit it is tempting to relate the parameter l to the inverse of the string tension α' by $l = \sqrt{\alpha'}$.

An interesting boundary phenomenon takes place [14] in the range $\eta < 0$. One can imagine heuristically, that the boundary develops a sort of attractive force, producing a boundary bound state. Indeed, in addition to the scattering states (5), the operator $-\partial_y^2$ admits for $\eta < 0$ the bound state

$$\psi_b(y) = \sqrt{2|\eta|} e^{\eta y}, \quad y \in \mathbb{R}_+. \quad (37)$$

The complete orthonormal set of eigenfunctions reads now $\{\psi(\lambda, y), \psi_b(y)\}$. As a consequence, the KK measure ϱ acquires [23] a δ -like contribution, localized at $M^2 - \eta^2$. The theory thus satisfies the spectral condition for $-M \leq \eta < 0$ and has a tachyon brane excitation for $\eta < -M$.

For definiteness we have considered in this paper $s = 1$ noncompact extra dimensions. Both the framework and the results, however, admit a direct generalization to $s \geq 1$. In that case $\lambda \mapsto (\lambda_1, \dots, \lambda_s)$, where λ_i has discrete spectrum if the associated dimension is compact.

Summarizing, we have demonstrated that the field dynamics on the brane is deeply influenced by the initial conditions in the bulk. The latter determine the KK measure, which is a fundamental ingredient of any theory with extra dimensions. Our analysis reveals also the existence of a large variety of quantum fields, generated by the same local action in the bulk, but corresponding to different initial conditions. These fields fall into three classes, having distinct locality properties. All of them preserve positivity, Poincaré invariance and the spectral condition (for $\eta > -M$) on \mathbf{M}_{3+1} . It turns out that gradually relaxing local commutativity, first in the bulk and after that on the brane, allows to derive renormalizable, and even finite, theories. Extra dimensions therefore emerge as a promising tool for constructing fundamental, and not only effective, 3+1 dimensional quantum field theories. In this respect, the idea of implementing a string-like nonlocality [24] on the brane, by imposing suitable nonlocal initial conditions in the bulk, looks very attractive. It will be interesting to extend our results to more realistic models, which involve gauge and gravitational interactions. Among others, one possibility will be to impose type B₁ and B₂ initial conditions in the gauge and gravitational sectors respectively, aiming to induce a unified renormalizable theory on the brane. Some variants of this scenario are currently under investigation [25].

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